Conjugated Heat Transfer from a Strip Heater with the Unsteady Surface Element Method

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The unsteady surface element method is extended to conjugated heat transfer from a strip heater where the unsteadiness is caused by a step change in the heater output flux. The geometry is a model of a flush-mounted, hot-film anemometer for air flow. The numerical results for the heater temperature agree with the analytical solution for early times and the steady-state results for the Nusselt number agree with an existing numerical solution.

Nomenclature

a = half length of heater, m
C = constant, Eq. (A3)
k = thermal conductivity, W/(mK)
L_j = length of jth surface element, m
M = number of timesteps
N = number of surface elements
N_β = conjugate Peclet number, Eq. (A4)
Nu = Nusselt number, Eq. (29)
Pe = $\beta a^2/\alpha_f$
p = heater output flux, W/m^2
q = heat flux, W/m^2
T = temperature, K
t = time, s
t^+ = $\alpha_s t/a^2$
x = streamwise coordinate
x^+ = x/a
x_j = center of jth surface element
y = transverse coordinate
α = thermal diffusivity, m^2/s
β = velocity gradient, s^{-1}
ϵ = defined by Eq. (25)
$\Gamma(\ )$ = gamma function
θ = Laplace variable
λ = dummy time variable
$\mu(\ )$ = unit step function
ξ = dummy space variable
ϕ = defined by Eq. (15)
$\tilde{\psi}$ = fundamental solution
$\Delta\psi$ = defined by Eq. (9)

Subscripts

av = spatial average over the heater
f = fluid
s = solid
k = kth timestep
0 = initial value

Superscripts

f = fluid
s = solid
$*$ = Laplace transform
$+$ = dimensionless variable

Introduction

THE unsteady surface element (USE) method is a powerful numerical technique for the solution of linear transient two- and three-dimensional heat transfer problems. It was developed by Keltner and Beck[1] to solve transient conduction problems with dissimilar bodies attached to one another over part of their boundaries. The method uses a Duhamel integral to express the problem as a single integral equation. This USE equation can be solved numerically and in limiting cases an analytical solution is possible.

In this paper the USE method is extended to transient conjugated heat transfer, the simultaneous flow of heat through a fluid and an adjacent solid. Conjugated heat transfer is important in thermal anemometry, in cooling of electronic components, in heat exchanger transients, and in other applications where the interface conditions between the fluid and the solid are not known. The particular problem to be considered is the transient temperature in a steady external flow due to a strip heater embedded in a semi-infinite wall. Refer to Fig. 1. The fluid is air and the wall is glass or quartz. This is the geometry of a commercially available flush-mounted, hot-film anemometer with a large aspect ratio. Initially the temperature is uniform at T_0 and the heater is turned on at time zero. The temperature far from the heater remains at T_0 for any finite time t.

Previous Work

The problem of conjugated heat transfer from a strip heater is related to the analysis of a flush-mounted, hot-film anemometer used for measuring wall shear stress. Most of the previous work has been for steady flow and steady heat transfer. Tanner[2] used a double Fourier-transform method to find the average temperature on a rectangular heater on a plane wall that is cooled by a shear flow. Tanner neglected streamwise and cross-stream heat conduction in the flow, and assumed that heat is generated uniformly over the heater. In the air/glass case, Tanner found that the conduction of heat into the glass has the most important effect on the heater temperature. Davies and Kimber[3] carried out a similar calculation on a two-dimensional strip heater, except that the streamwise conduction of heat in the fluid was retained. Brosh[4] used a finite-difference calculation to examine a two-dimensional problem with a line heat source at the wall. Brosh reported detailed temperature distributions and showed that a

Received April 9, 1986; presented as Paper 86-1242 at the AIAA/ASME 4th Thermophysics and Heat Transfer Conference, Boston, MA, June 2–4, 1986; revision received July 31, 1986.

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simple shear flow gave the same wall temperature distributions as a laminar velocity profile. Kalumuck,[5] in the most comprehensive steady-state calculation, considered the fully three-dimensional problem with a semi-infinite wall and a pure shear flow. Kalumuck used a Fourier-transform method with a numerical transform inversion. Kalumuck carried out a parametric study of the average heater temperature due to changes in the thermal conductivity ratio (k_f/k_s), the aspect ratio of the rectangular heater (b/a), and the Peclet number of the flow. Kalumuck also reported detailed temperature and heat flux distributions on the fluid/solid interface. Kalumuck found that for the case of air flow over a glass wall, the strip heater warms a large portion of the wall, and that most of the heat flux to the fluid takes place upstream of the heater.

There is little work on transient heat transfer for this geometry. Perelman[6] used integral equations to examine the heat transfer from a solid with an arbitrary heat source distribution cooled by a laminar compressible gas flow. Temperature gradients in the wall were neglected perpendicular to the flow direction (lumped wall). Perelman introduced the dimensionless parameter $B \equiv 2\alpha_s/(Re\alpha_f)$ where Re is a Reynolds number. Parameter B represents the relative importance of the transient responses of the fluid and the solid. When $B \ll 1$, as in the air/glass case, the fluid transient term in the describing differential equation can be neglected. The problem is quasisteady in the fluid and unsteady in the solid. Perelman does not report any numerical results.

Degani[7] used a finite-difference method to calculate a two-dimensional compressible flow over a compression corner with a heat source embedded in the wall. The effect of viscous dissipation and temperature-dependent thermal properties were included in the calculation. The thermal transient was calculated due to a single "pulse" of temperature at the heater location. Degani showed that the transient temperature strongly depends on the location of the heater relative to the separation bubble in the compression corner.

Numerical Methods

The usual numerical methods for transient heat transfer problems are the finite element (FE) and the finite difference (FD) methods. These methods are flexible and well-known, but they are not efficient in the type of problem mentioned previously for two reasons. First, it is necessary to set up very fine grids near the interface and to include many grid points far away from the interface. This results in very large systems of simultaneous algebraic equations, especially for two- and three-dimensional problems. Second, these methods generate the solution at every grid point in the domain, even if the solution is only needed at the interface.

A method similar to the USE method is the boundary element (BE) method. It is well-suited for solving problems with irregular boundaries or with infinite domain, and the method has been applied to transient problems, but the entire boundary must be discretized.

For conjugated heat transfer where the heating is confined to a small area, the USE method is superior to the FE, FD, and BE methods. In the USE method, only the "active interface" between the fluid and the solid must be discretized. The active interface is that part of the interface where the heat passing between the fluid and the solid is nonzero. This reduces the size of the numerical calculations and reduces computer time. In addition, the USE method first produces the transient temperature at the interface, and the temperature at internal points can be calculated later if needed.

USE Method

The USE method is (at present) limited to problems described by the linear heat conduction equation, which limits the present work to constant thermal properties and no buoyant flow effects (small temperature rise at the heater). In addition, the heater is very thin so that thermal storage can be neglected and the temperature is uniform across the heater thickness.

Duhamel Integral

The Duhamel integral is used to express the conjugated heat transfer problem in the form of integral equations. The derivation involves splitting the geometry in two and separately expressing the temperature in the solid and the fluid as integrals of an unknown interface heat flux distribution and a known "fundamental solution." The fundamental solutions can be related to Green's functions and they may be found from analytical expressions or from numerical calculations.

First consider the solid side. The unknown temperature in the solid is $T^s(x,y_s,t)$ and the unknown heat flux into the solid at the interface is $q^s(x,t)$. The fundamental solution, $\psi^s(x-\xi,y_s,t)$, is the temperature rise in the solid due to a unit increase in the boundary heat flux over the half-plane $x>\xi$ and insulated elsewhere. Refer to Fig. 2. It is described by the following equations for the homogeneous semi-infinite solid:

$$\frac{\partial \psi^s}{\partial t} = \alpha_s\left(\frac{\partial^2 \psi^s}{\partial x^2} + \frac{\partial^2 \psi^s}{\partial y_s^2}\right) \qquad \begin{array}{c} -\infty < x < \infty \\ 0 < y_s < \infty \end{array} \tag{1a}$$

$$\psi^s(x-\xi,y_s,t=0)=0 \tag{1b}$$

$$-k_s\frac{\partial \psi^s}{\partial y_s}(x-\xi,y_s=0,t) = \begin{cases} 0; & t<0 \text{ or } x<\xi \\ 1; & t>0 \text{ and } x>\xi \end{cases} \tag{1c}$$

The analytical expression for ψ^s is discussed in the Appendix. The Duhamel integral is used to give the temperature anywhere in the solid due to an arbitrary heat flux into the boundary as[8]

$$T^s(x,y_s,t) - T_0 =$$

$$-\int_{\xi=-\infty}^{\infty}\int_{\lambda=0}^{t} q^s(\xi,\lambda)\frac{\partial^2 \psi^s}{\partial \xi \partial t}(x-\xi,y_s,t-\lambda)\, d\xi d\lambda \tag{2}$$

where T_0 is the initial temperature. The integral on ξ is over the entire interface.

Next consider the fluid side. The fluid fundamental solution, $\psi^f(x-\xi,y_f,t)$, is the temperature rise in the fluid due to a

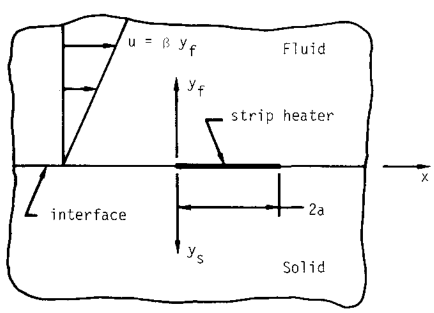

Fig. 1 Geometry for conjugated heat transfer.

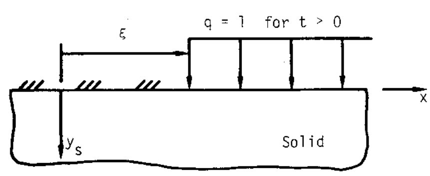

Fig. 2 Geometry for the solid fundamental solution.

unit increase in the boundary heat flux over the half-plane $x>\xi$ and insulated elsewhere. Refer to Fig. 3. The fluid fundamental solution is given by the energy equation for the forced convection of a constant-property fluid. The energy equation can be simplified in the particular case where the fluid is air, the wall is glass or quartz, the heater is on the order of 0.5 mm in length, and the temperature rise due to the heater is small. Perelman[6] showed that the fluid unsteady term can be neglected if the time-constant of the fluid thermal response is much smaller than that of the solid, and this is true for the air/glass system. Ling[9] computed the effects of streamwise conduction of heat in the fluid-only problem. Ling showed that when the streamwise conduction of heat is neglected, the heat transfer from the strip is in error by less than 10% for air flow. Then, because in the conjugated heat transfer problem less than 5% of the heater output goes directly to the fluid, the error due to neglecting this term is less than 1%. In a steady-state calculation, Brosh[4] showed that the same temperature field is found if the laminar external velocity profile is replaced by a simple shear velocity profile. Finally, because the heater is small in length the variation of shear stress along the interface is neglected. In this case the equation for the fluid fundamental solution, ψ^f, is

$$\beta y_f \frac{\partial \psi^f}{\partial x} = \alpha_f \frac{\partial^2 \psi^f}{\partial y_f^2} \tag{3a}$$

$$\psi^f(x-\xi, y_s, t=0) = 0 \tag{3b}$$

$$-k_f \frac{\partial \psi^f}{\partial y_f}(x-\xi, y_f = 0, t) = \begin{cases} 0; & t<0 \text{ or } x<\xi \\ 1; & t>0 \text{ and } x>\xi \end{cases} \tag{3c}$$

The analytical expression for ψ^f is discussed in the Appendix.

The temperature in the fluid, $T^f(x,y_f,t)$, due to an arbitrary heat flux distribution into the fluid, $q^f(x,t)$, is given by a Duhamel integral as

$$T^f(x,y_f,t) - T_0 = -\int_{-\infty}^{\infty}\int_0^t q^f(\xi,\lambda)\frac{\partial^2 \psi^f}{\partial \xi \partial t}(x-\xi, y_f, t-\lambda)\,\mathrm{d}\xi\,\mathrm{d}\lambda \tag{4}$$

Matching Conditions

The temperature matching condition at the interface is based on a very thin heater of high thermal conductivity (thin metal film) and perfect thermal contact among the fluid, the solid, and the heater. The fluid and solid temperatures simultaneously match at the interface:

$$T^f(x,y_f=0,t) = T^s(x,y_s=0,t) \tag{5}$$

The heat flux matching condition is given by an energy balance at the interface with zero thermal storage in the heater:

$$p(x,t) = q^s(x,t) + q^f(x,t) \tag{6}$$

where $p(x,t)$ is the heater output flux. At the heater, the output flux is split between the solid and the fluid. Away from the heater, where $p(x,t)=0$, the fluid and solid heat fluxes sum to zero.

Discretization of the Interface

The active interface is that part of the interface over which heat is transferred between the solid and the fluid. The active interface is a truncation of the actual infinite interface, but the error is small if the active interface is sufficiently large. The active interface is discretized into N surface elements of length L_j, $j=1,2,\ldots,N$, as shown in Fig. 4. Each surface element has

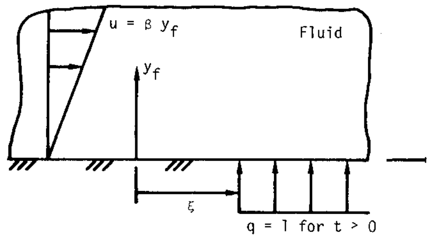

Fig. 3 Geometry for the fluid fundamental solution.

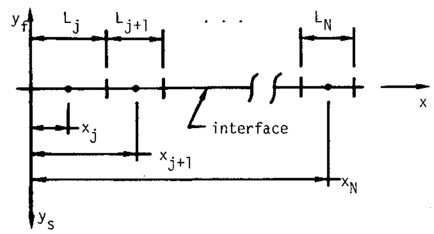

Fig. 4 Discretizations for the interface into surface elements.

a temperature and a heat flux associated with it. The temperature associated with element j is taken to be the temperature at the center of the surface element, located at point x_j. The heat flux associated with surface element j is taken to be spatially uniform over the element.

The Duhamel integral for the solid temperature, Eq. (2), can be written approximately as the sum of N integrals, one for each surface element. If x_j is the center of element j with length L_j, then the integral limits for each element are $(x_j - L_j/2)$ and $(x_j + L_j/2)$. The heat flux, which is constant over each surface element, can be taken outside each integral over x:

$$T^s(x,y_s,t) - T_0 = -\int_0^t \partial\lambda \sum_{j=1}^{N} \times \left[q^s(x_j,\lambda)\int_{xj-Lj/2}^{xj+Lj/2}\frac{\partial^2\psi^s}{\partial\xi\partial t}x-\xi,y_s,t-\lambda)\mathrm{d}\xi\right] \tag{7}$$

The integral over each surface element can be evaluated to give

$$T^s(x,y_s,t) - T_0 = -\sum_{j=1}^{N}\int_0^t \times q^s(x_j,\lambda)\frac{\partial}{\partial t}[\Delta\psi^s(x,x_j,y_s,t-\lambda)]\mathrm{d}\lambda \tag{8}$$

where $\Delta\psi^s$ is defined by

$$\Delta\psi^s(x,x_j,y_s,t) = \psi^s\left(x-\left(x_j+\frac{L_j}{2}\right),y_s,t\right) - \psi^s\left(x-\left(x_j-\frac{L_j}{2}\right),y_s,t\right) \tag{9}$$

The fluid Duhamel integral is developed in a similar fashion from Eq. (4) to give

$$T^f(x,y_f,t) - T_0 = -\sum_{j=1}^{N}\int_0^t \times q^f(x_j,\lambda)\frac{\partial}{\partial t}[\Delta\psi^f(x,x_j,y_f,t-\lambda)]\mathrm{d}\lambda \tag{10}$$

where $\Delta\psi^f$ is defined in a fashion analogous to Eq. (9).

The USE Equation

The fluid and solid integral equations and the matching conditions can now be combined to form a set of integral equations where the unknowns are the fluid heat flux histories, $q^f(x_j,t)$, $j=1,2,\ldots,N$. First, Eqs. (8) and (10) are evaluated at the points $x=x_i$, the centers of the surface elements. Then, Eqs. (6), (8), and (10) are substituted into the temperature matching condition, Eq. (5), to give

$$\sum_{j=1}^{N}\int_0^t q^f(x_j,\lambda)\frac{\partial}{\partial t}[\Delta\psi^f(x_i,x_j,0,t-\lambda)]d\lambda$$
$$=\sum_{j=1}^{N}\int_0^t [p(x_j,\lambda)-q^f(x_j,\lambda)]\frac{\partial}{\partial t}[\Delta\psi^s(x_i,x_j,0,t-\lambda)]d\lambda \qquad (11)$$

for $i=1,2,\ldots,N$. Because the ψ-functions and $p(x_j,t)$ are known, this set of N integral equations can be solved for the N heat flux histories, $q^f(x_j,t)$. Once the heat flux histories have been found, the temperature values at any location can be calculated with Eqs. (8) and (10).

Numerical Solution

In a numerical solution of the USE method the time integral in Eq. (11) is replaced by a suitable numerical representation. A simple discretization of the time span $(0,t)$ is discussed here. As the problem is linear in the heat flux, other numerical techniques are possible, such as the discrete Fourier transform.

Let the time interval $(0,t)$ be divided into M equal-sized time steps of length Δt so that t_k represents the time at the end of the kth time interval ($t_k=k\Delta t$). The heat flux histories $q^f(x_j,t)$ are assumed to have constant values within each time interval. The time integrals in Eq. (11) can be written as a sum of integrals over the timesteps as

$$\sum_{j=1}^{N}\sum_{k=1}^{M}\left\{q^f(x_j,t_k)\int_{t_{k-1}}^{t_k}\frac{\partial}{\partial t}[\Delta\psi^f(x_i,x_j,0,t_M-\lambda)]d\lambda\right\}$$
$$=\sum_{j=1}^{N}\sum_{k=1}^{M}\left\{[p(x_j,t_k)-q^f(x_j,t_k)]\right.$$
$$\left.\int_{t_{k-1}}^{t_k}\frac{\partial}{\partial t}[\Delta\psi^s(x_i,x_j,0,t_M-\lambda)]d\lambda\right\} \qquad (12)$$

The time integrals can be evaluated to give

$$\sum_{j=1}^{N}\sum_{k=1}^{M}\{q^f(x_j,t_k)[\Delta\psi^f(x_i,x_j,0,t_M-t_k)$$
$$-\Delta\psi^f(x_i,x_j,0,t_M-t_{k-1})]\}$$
$$=\sum_{j=1}^{N}\sum_{k=1}^{M}\{[p(x_j,t_k)-q^f(x_j,t_k)]$$
$$\times[\Delta\psi^s(x_i,x_j,0,t_M-t_k)-\Delta\psi^s(x_i,x_j,0,t_M-t_{k-1})]\} \qquad (13)$$

for $i=1,2,\ldots,N$. This represents N equations and they can be placed in matrix notation as

$$\sum_{k=1}^{M}(\underset{\sim}{\phi}^f_{M-k+1}-\underset{\sim}{\phi}^f_{M-k})\underset{\sim}{q}^f_k$$
$$=\sum_{k=1}^{M}(\underset{\sim}{\phi}^s_{M-k+1}-\underset{\sim}{\phi}^s_{M-k})(\underset{\sim}{p}_k-\underset{\sim}{q}^f_k) \qquad (14)$$

Table 1 Results for spatial average temperatures on the heater for $N_\beta=0.335$

t^+	Analytical, Eq. (28)	Numerical[a]	Percentage difference
0.003	0.06017	0.06040	0.38
0.012	0.1172	0.1177	0.43
0.030	0.1796	0.1807	0.61
0.060	0.2454	0.2465	0.45
0.090	0.2927	0.2945	0.61
0.120	0.3305	0.3329	0.73
0.150	0.3623	0.3653	0.83
0.180	0.3898	0.3933	0.90
0.210	0.4142	0.4182	0.97
0.240	0.4361	0.4406	1.03
0.285	0.4651	0.4703	1.12

[a]Nine surface elements, $k_f/k_s=0.024$, $\alpha_f/\alpha_s=65$, $a=0.25$ mm.

where $\underset{\sim}{\phi}^s_k$ and $\underset{\sim}{\phi}^f_k$ are matrices whose components are

$$(\underset{\sim}{\phi}^s_k)_{ij}=-\Delta\psi^s(x_i,x_j,0,t_k)>0 \qquad (15a)$$
$$(\underset{\sim}{\phi}^f_k)_{ij}=-\Delta\psi^f(x_i,x_j,0,t_k)>0 \qquad (15b)$$

and $\underset{\sim}{q}^f_k$ and $\underset{\sim}{p}_k$ are column vectors whose components are

$$\underset{\sim}{q}^f_k=[q^f(x_1,t_k),q^f(x_2,t_k),\ldots,q^f(x_N,t_k)]^T \qquad (15c)$$
$$\underset{\sim}{p}_k=[p(x_1,t_k),p(x_2,t_k),\ldots,p(x_N,t_k)]^T \qquad (15d)$$

Equation (14) can be written with the unknown vector $\underset{\sim}{q}^f_M$ on the left-hand side as

$$(\underset{\sim}{\phi}^s_1+\underset{\sim}{\phi}^f_1)\underset{\sim}{q}^f_M=\sum_{k=1}^{M-1}(\underset{\sim}{\phi}^s_{M-k+1}-\underset{\sim}{\phi}^s_{M-k})(\underset{\sim}{p}_k-\underset{\sim}{q}^f_k)$$
$$-\sum_{k=1}^{M-1}(\underset{\sim}{\phi}^f_{M-k+1}-\underset{\sim}{\phi}^f_{M-k})\underset{\sim}{q}^f_k+\underset{\sim}{\phi}^s_1\underset{\sim}{p}_M \qquad (16)$$

where $\underset{\sim}{\phi}^s_0=\underset{\sim}{\phi}^f_0=0$ has been used ($\underset{\sim}{0}$ is the zero matrix). Equation (16) has the form of a matrix equation $\underset{\sim}{A}\,\underset{\sim}{q}_M=\boldsymbol{B}$. It can be solved in the manner of Litkouhi[10] by beginning at the first timestep ($M=1$) and solving for $\underset{\sim}{q}^f_1$, then going to the next timestep ($M=2$) and solving for $\underset{\sim}{q}^f_2$, and so on.

Analytical Solution

The set of simultaneous integral equations represented by Eq. (11) can be solved with the Laplace transform technique in the case of a single surface element. Consider the problem of one surface element that corresponds to the heater location. This model is very useful for small times. Soon after the heater is turned on, heat flows from the heater normal to the interface to the fluid and the solid, and the interface that is not part of the heater does not participate in the heat transfer.

For a single surface element, Eq. (11) reduces to a single integral equation,

$$\int_0^{t^+} q(\lambda^+)\frac{\partial}{\partial t^+}[\Delta\psi^f(t^+-\lambda^+)]d\lambda^+$$
$$=\int_0^{t^+}[p(\lambda^+)-q(\lambda^+)]\frac{\partial}{\partial t^+}[\Delta\psi^s(t^+-\lambda^+)]d\lambda^+ \qquad (17)$$

where the fundamental solutions $\Delta\psi^f$ and $\Delta\psi^s$ are now the average temperature on the heater due to unit step heating. The x-dependence has been absorbed into this average. This equation can be Laplace transformed with the identity for

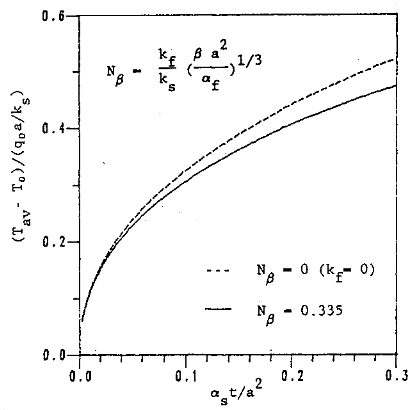

Fig. 5 Spatial average temperature on the heater versus time from the analytical solution.

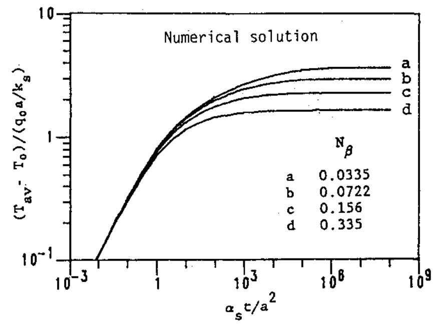

Fig. 6 Spatial average temperature on the heater versus time from the numerical solution.

convolution integrals to give

$$\theta q^*(\theta)[\Delta\psi^f(\theta)]^* = \theta[p^*(\theta) - q^*(\theta)][\Delta\psi^s(\theta)]^* \tag{18}$$

where θ is the dimensionless Laplace transform variable. The solution for $q^*(\theta)$ is

$$q^*(\theta) = \frac{p^*(\theta)[\Delta\psi^s(\theta)]^*}{[\Delta\psi^f(\theta)]^* + [\Delta\psi^s(\theta)]^*} \tag{19}$$

This expression can be inverted to get the heat flux to the fluid, but the temperature on the heater can be obtained directly. Equation (10), the fluid temperature, can be evaluated for the average temperature on the heater and Laplace transformed to give

$$T^*_{av}(\theta) = \theta q^*(\theta)[\Delta\psi^f(\theta)]^* \tag{20}$$

where $T^*_{av}(\theta)$ is the transform of the heater temperature. By replacing $q^*(\theta)$ with Eq. (19), the transform of the temperature on the heater is given by

$$T^*_{av}(\theta) = \frac{\theta p^*(\theta)[\Delta\psi^s(\theta)]^*[\Delta\psi^f(\theta)]^*}{[\Delta\psi^f(\theta)]^* + [\Delta\psi^s(\theta)]^*} \tag{21}$$

The same result is obtained if the equation for the temperature in the solid, Eq. (8), is used for the derivation. To complete the solution, the expression for $T^*_{av}(\theta)$ must be inverted into the time domain. The ease or difficulty of the inversion process depends entirely on the particular expressions that go into $T^*_{av}(\theta)$.

The fundamental solutions $\Delta\psi^f$ and $\Delta\psi^s$ are given in the Appendix in Eqs. (A1) and (A3). Their Laplace transforms are

$$\Delta\psi^s(\theta)^* = \frac{a}{k_s\theta}\left(\frac{1}{\sqrt{\theta}} - \frac{1}{\pi\theta}\right) \tag{22}$$

Table 2 Steady-state Nusselt number as a function of Pe and k_f/k_s

N_β	Pe	k_f/k_s	Kalumuck (1983)	USE method	Percentage difference
0.317	4	0.2	5.94	5.82	1.9
0.504	16	0.2	7.44	7.35	1.2
0.660	36	0.2	8.63	8.55	0.93
0.8	64	0.2	9.66	9.59	0.71
0.159	4	0.1	8.79	8.69	1.2
0.252	16	0.1	10.98	10.49	0.83
0.330	36	0.1	11.94	11.87	0.62
0.4	64	0.1	13.10	13.03	0.53
0.0317	4	0.02	—	25.96	—
0.0504	16	0.02	30.04	29.64	1.3
0.0600	36	0.02	—	32.15	—
0.08	64	0.02	34.41	34.16	0.73

$$\Delta\psi^f(\theta)^* = \frac{a}{k_s CN_\beta\theta} \tag{23}$$

A step input for the heater is given by $p(t) = q_0\mu(t)$, where $\mu(t)$ is the unit step function, and the Laplace transform of this step input is $p^*(\theta) = q_0/\theta$. Place these expressions into Eq. (21) to get

$$T^*_{av}(\theta) = q_0\frac{a}{k_s}\left(\frac{1}{\sqrt{\theta}} - \frac{1}{\pi\theta}\right)\Bigg/\theta\left[1 + CN_\beta\left(\frac{1}{\sqrt{\theta}} - \frac{1}{\pi\theta}\right)\right] \tag{24}$$

To inverse transform this expression for small times, use the fact that $1/\theta \ll 1$ when t is small. Let

$$\epsilon = CN_\beta\left(\frac{1}{\sqrt{\theta}} - \frac{1}{\pi\theta}\right) \tag{25}$$

so that

$$T^*_{av}(\theta) = \frac{q_0 a}{\theta k_s CN_\beta}\left(\frac{\epsilon}{1+\epsilon}\right) \tag{26}$$

Because $1/\theta$ is small and because $CN_\beta < 1$ for the air/glass case, then ϵ is also small. The ratio $1/(1+\epsilon)$ can be replaced by the binomial expansion to give

$$T^*_{av}(\theta) = \frac{q_0 a\epsilon(1 - \epsilon + \epsilon^2 - \epsilon^3 + \ldots)}{\theta k_s CN_\beta} \tag{27}$$

When multiplied out, the resulting expression can be inverse transformed term-by-term.The final result for retaining terms up to $1/\theta^3$ is

$$T^+_{av}(t^+) = \frac{T_{av} - T_0}{q_0 a/k_s} = 2(t^+/\pi)^{1/2} - t^+(1/\pi + CN_\beta)$$
$$+ \frac{4}{3\pi}(t^+)^{3/2}[2CN_\beta/\pi + (CN_\beta)^2]$$
$$- \tfrac{1}{2}(t^+)^2[CN_\beta/\pi^2 + 3(CN_\beta)^2/\pi + (CN_\beta)^3] \tag{28}$$

for $t^+ < 0.3$. In the limit as $N_\beta \to 0$ (no fluid) this expression reduces to the solid fundamental solution.

The steady state is reached only after a very long time. For $N_\beta = 0.335$ with a glass wall, steady state is reached after about 10^4 s (several hours). This points out a limitation in the semi-infinite wall geometry for modeling a hot-film anemometer, because in an actual anemometer, heat leakage paths through electrical wires or through the back of the sensor shortens the time required to reach steady state.

A Nusselt number can be defined with the average heater temperature T_{av} and the output flux from the heater q_0 as

$$Nu = \frac{q_0(2a)}{(T_{av} - T_0)k_f} = \frac{2k_s}{T_{av}^+ k_f} \tag{29}$$

Kalumuck[5] has calculated steady-state Nusselt numbers and a comparison with the present calculations are shown in Table 2. Kalumuck used a spatial Fourier transform method with a numerical transform inversion. The USE method steady-state results are calculated with one large time step. Kalumuck includes streamwise conduction of heat in the fluid and the present calculation does not. In Table 2, the parameter N_β has been split into two parameters Pe and k_f/k_s because the Nusselt number explicitly depends on both Pe and k_f/k_s. Table 2 shows a difference of less than 2% in the results for $Pe = 16$ and $k_f/k_s = 0.2$. This difference decreases as Pe increases and as k_f/k_s decreases. In the $k_f/k_s = 0.02$ case (approximately the air/glass case), there is a break in the trend in decreasing difference as k_f/k_s decreases, but Kalumuck reports a $\pm 1\%$ uncertainty due to numerical precision in this case. The accuracy for the present work is about 0.3% based on studies of the effect of the discretization parameters Δt and L_j. For the air/glass case, these results confirm that the fluid streamwise heat conduction can be neglected, but for the water/glass case ($k_f/k_s \approx 0.4$) this effect cannot be neglected.

Summary and Conclusions

The unsteady surface element (USE) method has been applied to the transient conjugated heat transfer from a strip heater. The work was made possible by the choice of a realistic fluid fundamental solution. The analytical and numerical USE method results closely agree for small times for a step change in the heater output. The numerical results have been presented for a wide range of time, and the steady-state results are in good agreement with a previous numerical solution. The steady-state USE results were obtained with one time step.

This research is a first step toward a transient model of a hot-film anemometer for the measurement of transient wall shear stress in air flow. A next step will be to replace the semi-infinite wall with a finite wall to better model the heat leakage from actual hot-film geometries.

Results and Discussion

The conjugated heat transfer problem is described by a single parameter N_β. That is, the interface temperature is given by $T^+ = T^+(x^+, t^+, N_\beta)$. For laminar flow of air over a glass wall the range of parameter values is $0.005 < N_\beta < 0.5$. The lower limit is related to avoiding natural convection and the upper limit is related to avoiding turbulent flow. The case $N_\beta = 0$ corresponds to the insulated solid (zero heat flow to the fluid).

A plot of the spatial average temperature on the heater from the analytical solution is shown in Fig. 5. The results are for air flow over glass with $a = 0.25$ mm. The analytical solution, Eq. (28), is most accurate near $t^+ = 0$ and it is only useful up to $t^+ = 0.3$. The analytical solution is compared to the numerical solution in Table 1. The two methods agree within 1.1% for the $N_\beta = 0.335$ case. The numerical solution used nine surface elements on the heater with smaller surface elements near the edges of the heater.

In Fig. 6 the numerical results for the spatial average temperature on the heater are plotted for various values of N_β. For t^+ small all of the curves lie together on the curve $T_{av}^+ \sim (t^+)^{1/2}$. This curve corresponds to the surface temperature for one-dimensional heat conduction, because heat conduction dominates for small times. Only at later times does the effect of the flow become visible, and as expected a larger velocity gradient gives a lower temperature. Typical computation time for one of the curves in Fig. 6 is 250 CPU seconds on a Prime 750 computer.

Appendix

Solid Fundamental Solution

The solid fundamental solution described by Eq. (1), $\psi^s(x, y_s, t)$, is given by Litkouhi.[10] The surface temperature $\psi^s(x, y_s = 0, t)$ needed in the numerical solution is given by Carslaw and Jaeger[11] on page 264. The appropriate solid fundamental solution for the analytical solution is the spatial average over the heated region. It is given by Litkouhi and Beck[12] and is approximated at early times by

$$\Delta\psi^s(t^+) = \frac{a}{k_s}(2(t^+/\pi)^{1/2} - t^+/\pi) \tag{A1}$$

for error less than 0.033% for $t^+ < 0.3$. The first term is the surface temperature for one-dimensional heat conduction into a semi-infinite body. The second term is an "edge correction" for the surface temperature on the heated strip at early time.

Fluid Fundamental Solution

The fluid fundamental solution described by Eq. (3), $\psi^f(x, y_f, t)$, is the temperature in a thermal boundary layer driven by a heat flux boundary condition. Most of the published theory on thermal boundary layers deals with temperature boundary conditions. A discussion of the fluid fundamental solution is given by Cole.[13] For the present work only the surface temperature $\psi^f(x, y_f = 0, t)$ is needed. It is given by Soliman and Chambré[14]:

$$\Delta\psi^f(x, y_f = 0, t) = \frac{9^{1/3}}{\Gamma 2/3} \frac{a}{k_s} \frac{k_s}{k_f} \left(\frac{\alpha_f}{\beta a^2}\right)^{1/3} (x/a)^{1/3} \mu(t)\mu(x) \tag{A2}$$

The average surface temperature over the heated region is needed for the analytical solution. It is given by

$$\frac{1}{2a}\int_0^{2a} \Delta\psi^f(x, y_f = 0, t)\,dx = \frac{a}{k_s C N_\beta}\mu(t) \tag{A3}$$

where

$$N_\beta = \frac{k_f}{k_s}\left(\frac{\beta a^2}{\alpha_f}\right)^{1/3} \tag{A4}$$

and $C = (2/3)^{5/3}\Gamma(2/3) = 0.6889$. The dimensionless parameter N_β contains the conductivity ratio k_f/k_s and a local Peclet number ($\beta a\, a/\alpha_f$), so the name conjugate Peclet number is suggested for N_β.

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